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Interaction of spiral waves with external fields in excitable media

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The problem of spiral waves interacting with weak external fields in two-dimensional reaction-diffusion excitable media is considered. The velocity and the angle of the drift of the spiral resulting from the interaction with the field is calculated analytically. We find that a field coupled to the fast variable causes the spiral to drift at an angle with respect to the direction of the field, whereas a field coupled to the slow variable induces a drift parallel to the field. Numerical simulations are presented that demonstrate a good qualitative agreement with the theory.

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Spiral waves, spontaneously formed from generic initial conditions, play an important role in the dynamics of two-dimensional (2D) active media, such as Belousov-Zhabotinsky reactions [1], cardiac tissues [2], some types of surface reactions [3], etc. In particular, spiral waves are responsible for such dangerous cardiac arrhythmias as fibrillation and ventricular tachycardias [4]. A vast body of research has been devoted to various methods of control and consequent annihilation of spiral waves [5–7]. The main approach employed is the application of an external field. The field causes the spiral to drift towards the boundaries where it is annihilated. Recent studies [5,8] have revealed a striking phenomenon—the spiral drifts at a significant angle with respect to the applied field. This phenomenon is reminiscent of the drift of the superconducting vortex at some angle to the applied current in type-II superconductors under the Lorentz force (see, e.g., [9]). Although from a theoretical point of view, one always should expect some nonzero angle of the drift because the spiral breaks both rotation and translation invariance of the media, there has been no quantitative theory relating the angle of the drift to the parameters of the active media.

The theory of wave propagation in excitable media in the presence of external fields can be described by the "reaction-diffusion" equations [10,11]

$$\partial_t u = \epsilon\nabla^2 u + [f(u,v)/\epsilon] - \epsilon\mathbf{E}_u \cdot \boldsymbol{\nabla} u \ ,$$
$$\partial_t v = \sigma\epsilon\nabla^2 v + g(u,v) - \sigma\epsilon\mathbf{E}_v \cdot \boldsymbol{\nabla} v \ , \qquad (1)$$

where $\epsilon \ll 1$ is a small positive parameter; u, v are the "fast" and "slow" variables, and $\mathbf{E}_u$, $\mathbf{E}_v$ their applied external fields (which we suppose to be weak), respectively; $\sigma = D_v/D_u$ is the ratio of diffusion coefficients of the two variables. In this paper we present a theory relating the angle and the velocity of a spiral's drift to the parameters of Eqs. (1). We consider the case of small fields in order to be able to calculate the effect of the fields as a perturbation of the isolated spiral wave. Nonlinear effects are discussed briefly at the end of the paper. The particular scaling of the applied fields in Eqs. (1) is motivated by the fact that external fields create additional flows of concentrations u,v and, therefore, should have the same mobility coefficients as diffusion flows. We have uncovered two different types of behavior depending on the ratio E_u/E_v. For the first case, which we call u controlled, defined by the condition $E_u, E_v \sim O(1)$, the spiral drifts at some nonzero angle to the applied field. For the case of a dominant slow variable field $E_v/E_u \sim 1/\sigma \gg 1$ (v-controlled case), the spiral drifts parallel to the field. Our results are in qualitative and quantitative agreement with numerical simulations of Eqs. (1).

Our solution of the problem is based on a generalization of the method [12,13] used in the case of inhomogeneous media. The procedure consists of two steps. In the first step the dynamics of the continuous two-dimensional active media is reduced to the dynamics of lines (*interfaces*) separating excited and quiescent regions. The dynamics of the interfaces is governed by the Gibbs-Thomson condition (see [10,11]). A special treatment is required in the core region, where the interfaces come together. The solution is significantly simplified in the so-called Fife scaling, valid for $\varepsilon^{1/3} \ll 1$. Solution of this problem gives an unperturbed spiral wave, and fixes the frequency of the spiral's rotation (for details see [14]). At the second step, the moving spiral is treated in linear order in the external field. In principle, linearization can be avoided but then the calculation becomes very cumbersome. Boundedness of the linear response fixes the spiral's velocity and the drift angle. We proceed from the second step, the first step being identical to Ref. [14].

In the presence of the external field E_u, the Gibbs-Thomson equation describing interfacial dynamics [10,11] is of the form

$$c_n = c(v) - \epsilon k + \epsilon \mathbf{n}_I \cdot \mathbf{E}_u \ , \qquad (2)$$

where $c(v)$ is the unperturbed interfacial velocity of the 1D case, c_n is the velocity normal to the interface, k is the local curvature of the interface, and $\mathbf{n}_I$ is the interfacial outer normal $\mathbf{n}_I \propto \boldsymbol{\nabla}[\theta - \theta_I(r)]$, (θ, r) are polar coordinates, $\theta_I(r)$ is the interfacial angle. The last term in Eq. (2) describes advection of the interface by the applied field.

Outside the core (outer region), in the frame corotating with the interface at frequency ω the Fife ansatz [15]

$$v - v_s = \epsilon^{1/3}\tilde{v} \ , \quad x = \epsilon^{2/3}\tilde{x} \ ,$$
$$t = \epsilon^{1/3}\tilde{t} \ , \quad c(v_I) \approx \epsilon^{1/3} c_v \tilde{v}_I \ , \quad \omega = \epsilon^{-1/3}\tilde{\omega} \ , \qquad (3)$$

where $c_v \equiv \frac{dc(v)}{dv}|_{v=v_s}$ [v_s and c_v are constants defined by

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the particular functions $f(u,v)$ and $g(u,v)$ in Eqs. 1)], together with the rescaling of the fields

$$\mathbf{E}_u = \epsilon^{-2/3}\mathbf{e}_u \ , \quad \mathbf{E}_v = \epsilon^{-2/3}\mathbf{e}_v \ , \tag{4}$$

brings Eqs. (1) and the interfacial equation (2) to the form

$$\partial_t v^{\pm} - \omega\partial_\theta v^{\pm} = g^{\pm} + \sigma\Delta v^{\pm} - \sigma \mathbf{e}_v \cdot \boldsymbol{\nabla} v^{\pm} \ , \tag{5}$$

$$c_n = c_v v^{\pm} - k + \mathbf{n}_I \cdot \mathbf{e}_u \ , \tag{6}$$

where the "+" and "−" signs correspond to the excited and quiescent regions, respectively. $g^{\pm} \equiv g[u^{\pm}(v_s), v_s]$=const. For an interface drifting with velocity $\mathbf{c_d} = (c_x, c_y)$ and rotating with frequency ω, the normal velocity c_n and curvature k are given by the expressions (with "+" and "−" corresponding to the front and back interfaces, respectively)

$$c_n^{\pm} = \pm\ (r\partial_t\theta^{\pm}/\sqrt{1+\psi^2}) + \mathbf{n}_I^{\pm} \cdot \mathbf{c}_d \ ,$$

$$k^{\pm} = \mp\ [1/(1+\psi^2)^{3/2}](d\psi/dr)\ \mp\ (\psi/r\sqrt{1+\psi^2}) \ , \tag{7}$$

where $\psi(r) \equiv r d\theta_I/dr$ and $\theta^{\pm} = \theta_0^{\pm}(r) - \theta_0^{+}(0) - \omega t + \delta\theta^{\pm}(r)\exp[i\omega t]+$ c.c. The term $\sigma\Delta v$ can be omitted everywhere except in the core region. It can be seen from Eq. (4) that the limitation upon the fields E_u, E_v to be "small enough" in the Fife limit, in order to consider them as a small perturbation is that $E_u, E_v \ll \epsilon^{-2/3}$.

In the u-controlled case the fields $\mathbf{E}_u$ and $\mathbf{E}_v$ both scale with σ as σ^0, i.e., $E_u \sim E_v \sim O(1)$. The term $e_v \cdot \nabla v$ can then be omitted in the outer region as well as in the core region.

In the system drifting with velocity $\mathbf{c_d}$ and rotating with frequency ω, Eq. (5) takes the form

$$\partial_t v^{\pm} - \omega\partial_\theta v^{\pm} = g^{\pm} + \mathbf{c_d} \cdot \boldsymbol{\nabla} v \ . \tag{8}$$

To linear order in c_d, one may replace v in the right-hand side of Eq. (8) by the unperturbed solution v_0 in the corotating frame $v_0^{\pm} = (-g^{\pm}/\omega)[\theta - \theta_0^{\pm}(r)]+$ const with $\theta_0^{+} - \theta_0^{-} = \Delta\theta_0 = 2\pi g^{-}/(g^{-} - g^{+})$. Without loss of generality, we may choose $g^{+} - g^{-} = 1$. In our rotating frame, the perturbation induced by the steady drift is explicitly time dependent and can be written in the form

$$\mathbf{c_d} \cdot \boldsymbol{\nabla} v_0 = -\hat{C}\frac{g^{\pm}(i-\psi)}{2\rho}\exp\{i[\omega t + \theta(r) - \theta^{+}(0)]\} + \text{c.c.} \tag{9}$$

where $\hat{C} = (c_x - ic_y)/\sqrt{\omega}$. We have introduced here a new variable $\rho = \sqrt{\omega}r$.

Representing, therefore, the solution of Eq. (8) in the form $v = v_0 + \delta v \exp[i\omega t]+$ c.c., we obtain (according to [12,13])

$$\delta v^{\pm}(\rho) = \left[A^{\pm}(\rho) + \hat{C}\theta\frac{g^{\pm}(i-\psi)}{2\omega\rho}\right]\exp\{i[\theta(r) - \theta^{+}(0)]\} - (g^{\pm}/\omega)\delta\theta(\rho) \ . \tag{10}$$

where $A^{\pm}$ are some unknown functions of ρ.

To make the perturbations $\delta v^{\pm}$ continuous across the interfaces, one needs to impose continuity conditions (see also [13,14]): $v^{+}[\theta_I^{+}(\rho)] = v^{-}[\theta_I^{+}(\rho)]$ and $v^{+}[\theta_I^{-}(\rho)] = v^{-}[2\pi + \theta_I^{-}(\rho)]$. Since $v_0^{\pm}$ satisfy unperturbed continuity conditions we obtain after simple algebra the relation $\delta\theta^{+} = \delta\theta^{-}\exp[i\Delta\theta]$.

In the corotating frame the normal projection of the external field oscillates with frequency ω. The normal projection of $\mathbf{e_u}$ is of the form

$$\mathbf{n}_I^{+} \cdot \mathbf{e}_u = \pm[(-\psi\mathbf{n}_\rho^{\pm} + \mathbf{n}_\theta^{\pm})/\sqrt{1+\psi^2}] \cdot \mathbf{e}_u = \hat{e}_u\sqrt{\omega}\frac{i-\psi}{2\sqrt{1+\psi^2}}\exp\{i[\theta^{+}(\rho) - \theta^{+}(0) + \omega t]\} + \text{c.c.}, \tag{11}$$

where $\hat{e}_u \equiv (e_u^x - ie_u^y)/\sqrt{\omega}$. Expanding ψ into an unperturbed part plus perturbation, $\psi = \psi_0 + \delta\psi$, we obtain from Eq. (7) that perturbations to the interfacial normal velocity and curvature satisfy the following conditions: $\delta c_n^{+} = -\delta c_n^{-}\exp[i\Delta\theta]$ and $\delta k^{+} = -\delta k^{-}\exp[i\Delta\theta]$ (with + and − signs corresponding to the front and back interfaces, respectively). This gives us an additional relation

$$\delta v_I^{+} = -\delta v_I^{-} e^{i\Delta\theta} \ . \tag{12}$$

Now, combining the contributions from the $\exp[i\omega t]$ harmonic of the perturbation, from the linearized equation (6) we obtain a closed equation for $\delta\theta^{+}$ [here we have used the δc_n and δk calculated in Eq. (7), together with (12)]:

$$\frac{\partial_\rho^2\delta\theta^{+} + (\frac{2}{\rho} + \rho\psi - \frac{3\psi\psi'}{1+\psi^2})\partial_\rho\delta\theta^{+}}{1+\psi^2} - i\delta\theta^{+} = f(\rho) \ , \tag{13}$$

where

$$f(\rho) = [(i-\psi)/2\rho][\hat{C} - \hat{e}_u + \hat{C}(B/\rho)\sqrt{1+\psi^2}] \times \exp\{i[\theta^{+} - \theta^{+}(0)]\} \ , \tag{14}$$

where B=const=1.738, ... is a universal constant coming from the solution of the unperturbed problem (see for instance, Refs. [10,16–18]).

Following the lines of Ref. [13] one can solve Eq. (13) analytically using its explicit homogeneous solutions. One homogeneous solution is given by the translation eigenmode

$$\chi_1 = [(i-\psi)/\rho]\exp[i\theta^{+}] \tag{15}$$

and the second solution can be obtained via reduction of order of Eq. (13)

$$\chi_2 = \chi_1\int_0^\rho d\rho'\frac{W(\rho')}{\chi_1^2} \ , \tag{16}$$

where the Wronskian $W(\rho) = \exp(-\int_0^\rho d\rho'\rho'\psi)\rho^{-2}(1+\psi^2)^{3/2}$. These solutions have the following asymptotic behaviors: $\chi_1 \to 1/\rho$, $\chi_2 \to$ const for $\rho \to 0$ and χ_1 is bounded, $\chi_2 \sim \exp[\rho^3/(3B)]$ for $\rho \gg 1$. The general solution of Eq. (13) can be represented in the form

$$\delta\theta^{+} = A_1\chi_1 + A_2\chi_2 + Z(\rho) \ , \tag{17}$$

where $Z(\rho)$ is the inhomogeneous solution bounded for $\rho \to \infty$:

$$Z(\rho) = -\frac{\chi_1}{2}\int_0^r d\rho' \chi_2(i-\psi)\left[\frac{\rho'}{\sqrt{1+\psi^2}}(\hat{C}-\hat{e}_u)+B\hat{C}\right]\exp\left[\int_0^{\rho'}\rho''\psi d\rho'' + i[\theta^+(\rho')-\theta^+(0)]\right]$$
$$-\frac{\chi_2}{2}\int_\rho^\infty d\rho' \chi_1(i-\psi)\left[\frac{\rho'}{\sqrt{1+\psi^2}}(\hat{C}-\hat{e}_u)+B\hat{C}\right]\exp\left[\int_0^{\rho'}\rho''\psi d\rho'' + i[\theta^+(\rho')-\theta^+(0)]\right]. \quad (18)$$

$Z(\rho)$ has the following asymptotic behavior as $\rho \to 0$:

$$Z(\rho) \;\to\; iz_1(\hat{C}-\hat{e}_u) \;+\; [(iB/2)\ln\rho + z_2]\hat{C} \quad (19)$$

and is bounded for large ρ. For our purpose we need an explicit expression for the constant z_1:

$$z_1 \;=\; \frac{1}{2}\int_0^\infty d\rho' \frac{(i-\psi)^2}{\sqrt{1+\psi^2}}\exp\left[\int_0^{\rho'}\rho''\psi d\rho'' + 2i[\theta^+(\rho')-\theta^+(0)]\right]. \quad (20)$$

z_1 is a universal model-independent complex constant and can be calculated numerically. The result is $z_1 = |z_1|\exp[i\eta]$ where $|z_1| \approx 0.576$ and $\eta \approx 0.8002 \approx 0.255\pi$. The constant z_2 is irrelevant for our analysis.

The constants $A_{1,2}$ are fixed by the asymptotic conditions at $\rho \to \infty$ and at the core $\rho \to 0$. The perturbative nature of the general solution (17) requires $A_2 = 0$ in order to rule out divergence of χ_2 as $\rho \to \infty$. We then have for the asymptotics of the outer solution at $\rho \to 0$ [our particular choice of the interfacial angles near origin is $\theta^\pm(0) = \pm\Delta\theta/2$]

$$\delta\theta^+ = (\hat{A}_1/\rho) \;-\; i\hat{A}_1 B \;-\; iz_1\hat{e}_u + \hat{C}[(iB/2)\ln\rho + iz_1 + z_2] + O(\rho)\,. \quad (21)$$

where $\hat{A}_1 = iA_1\exp(i\Delta\theta/2)$. The value of constant A_1 can be defined by matching with the inner (core) solution.

The core problem for the case of external field is much like that discussed in Refs. [13,14]. In order to fix $\hat{C}$ one has to find the perturbation of the interface $\delta\theta^+$ in the core. By virtue of linearity of the core problem, this perturbation has the outer asymptotic behavior for large distances (on the core scale) $\delta\theta^+ = \gamma\hat{C}$, both for the case of small diffusion σ and for the diffusionless case (see [13]). This has to be matched onto the inner asymptotics of the outer solution (21), which yields $\hat{C} = -iz_1\hat{e}_u/\gamma$. On the other hand, the outer asymptotics of the interfacial angles in the core should be the same as those in the outer problem near the origin. This gives us that $\arg(\gamma) = \Delta\theta/2 - \pi/2$. Then, choosing the X axis parallel to the field $\mathbf{E}_u$ ($E_u^y = 0$) and using the calculated z_1, we obtain

$$c_x = (E_u|z_1|/|\gamma|)\;\cos[(\Delta\theta/2)-\eta]$$
$$c_y = (E_u|z_1|/|\gamma|)\;\sin[(\Delta\theta/2)-\eta]\,. \quad (22)$$

Equations (22) are valid for the case $1/2 \le g^+ \le 1$ ($0 \le \Delta\theta \le \pi$). It follows from the symmetry of the core solution under $g^+ \leftrightarrow 1-g^+$, that $S_2^+(g^+) = -S_2^+(1-g^+)$ and $\Delta\theta \leftrightarrow 2\pi - \Delta\theta$. This results from the fact that mapping $g^+ \to 1-g^+$ simply interchanges the roles of the excited and quiescent regions. We have from Eqs. (22) that the angle ϕ between the drift direction and the field $\mathbf{E}_u$ is

$$\phi = \Delta\theta/2 - \eta \;=\; \pi|g^-| - \eta\,. \quad (23)$$

If $g^+ \le 1/2$ ($\Delta\theta \ge \pi$) then $\phi = \pi - \Delta\theta/2 - \eta$. Therefore, ϕ lies in the interval $-\eta \le \phi \le \pi/2 - \eta$.

This general calculation of the drift can be illustrated by the particular case of the small diffusion core, discussed in Refs. [13,14], where equations for the core problem have been obtained. One has to perform the so-called "Bernoff scaling" [14,17,18]: $\tilde{r} = r\sigma^{-1/3}$, $\tilde{v} = v\sigma^{1/3}$. The asymptotic solution of the core problem is given by [13]

$$\delta\theta^\pm = \sigma^{-2/3}\hat{C}[(\sigma^{1/3}S_1^\pm/\sqrt{\omega}\rho) + S_2^\pm + \cdots]\,. \quad (24)$$

The asymptotic constants $|S^\pm|$ have previously been found numerically from the solution of the core problem [13].

Comparing Eqs. (21) and (24) one can achieve the necessary matching if the relations $\hat{A}_1 = \sigma^{-1/3}S_1^+\hat{C}$ and $-iz_1\hat{E}_u = \sigma^{-2/3}S_2^+\hat{C}$ hold. Matching with the $\ln(\rho)$ term is achieved at higher order in $\sigma^{1/3}$. Therefore, the drift is given by the expression

$$\hat{C} = \;-\,i(\sigma^{2/3}z_1\hat{e}_u/S_2^+)\,. \quad (25)$$

As argued above, $\arg(S_2^+) = \Delta\theta/2 - \pi/2$. We then have

$$c_x = \sigma^{2/3}\epsilon^{2/3}(E_u|z_1|/2|S_2^+|)\;\cos[(\Delta\theta/2)-\eta]\,,$$
$$c_y = \sigma^{2/3}\epsilon^{2/3}(E_u|z_1|/2|S_2^+|)\;\sin[(\Delta\theta/2)-\eta]\,. \quad (26)$$

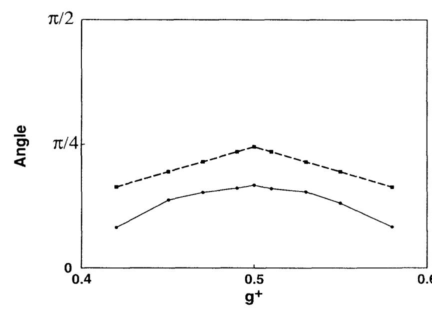

FIG. 1. Angle between the drift and the field $\mathbf{E}_u$ versus model parameter g^+, obtained in simulations (solid line) and predicted by theory (dashed line). The parameters of the simulations according to the model of Ref. [19] are $a = 2(1-g^+ + b), b = 0.01, \epsilon = 0.002, \sigma = 0.01$, the domain size is 8×8, number of the grid points is 121×121. The field amplitude $E_u = 0.0038$. To the left and to the right of the presented graph is the onset of the core meander instability. Here the further maximal error of determination of the angle is 3%.

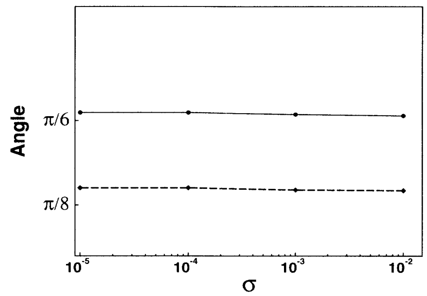

FIG. 2. Angle between the drift and the field $\mathbf{E}_u$ vs diffusion coefficient σ obtained in simulations for the symmetric spiral, $g^+ = 0.5$ ($a = 1.02$), and asymmetric spiral, $g^+ = 0.55$ ($a = 0.92$), given by the solid line and the dashed line, respectively. Other parameters are the same as in Fig. 1.

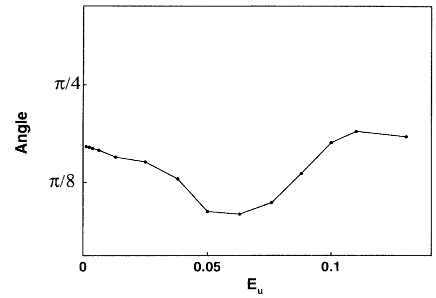

FIG. 3. Angle between the drift and the field $\mathbf{E}_u$ as a function of the field amplitude E_u, obtained in simulations for the symmetric spiral, $g^+ = 0.5$ ($a = 1.02$). Other parameters are the same as in Fig. 1.

Let us now turn to the v-controlled case. Then one has to take into account the term $\sigma \mathbf{e}_v \cdot \boldsymbol{\nabla} v$ in Eq. (5), so that $\mathbf{c}_d$ and $\hat{C}$ are replaced by

$$\mathbf{c}_d' = \mathbf{c}_d - \sigma\epsilon^{2/3}\mathbf{E}_v, \quad \hat{C}' = \hat{C} - \epsilon^{2/3}\hat{E}_v \ , \tag{27}$$

where

$$\hat{E}_v \equiv \sigma[(E_v^x - iE_v^y)/\sqrt{\omega}] \ \sim O(1) \ . \tag{28}$$

In that case one obtains, matching the outer and the core solution at first order in σ:

$$\hat{C} = \epsilon^{2/3}\hat{E}_v \ , \tag{29}$$

which means that the drift is always parallel to the field $\mathbf{E}_v$ up to terms higher orders in σ, with the absolute value $c_d = \sigma\epsilon^{2/3}E_v$. This result is in excellent agreement with the numerical simulations.

We have simulated the problem (1) by the EZ-spiral package written by Barkley [19] for the model given by $f(u,v) = u(u-1)[u - u_{th}(v)]$ and $g(u,v) = u - v$, where $u_{th}(v) = (v+b)/a$. It is a simple exercise to find $g^{\pm}$ for such a model. The result is $g^+ = 1 - a/2 + b$ and $g^- = g^+ - 1 = -a/2 + b$. We have added an external field $\mathbf{E}_u$ to EZ spiral and measured the angle between the resulting drift and the field as a function of g^+. In Fig. 1 the angle given by Eq. (23) is compared to the results of the simulations. We can see a good qualitative agreement between the effect of the angle predicted theoretically and observed in simulations. The small discrepancy is related presumably to not too small values of ϵ. Further decrease of ϵ is an extremely expensive project, in terms of CPU time.

Note that one cannot sweep all of the predicted angle range, because of the onset of the core meander instability. It should be remarked that there is as yet no satisfactory understanding of the core dynamics in the case of small ϵ, σ considered herein. Analytical treatments [14] indicate a real instability whereas numerical simulations show a Hopf bifurcation to meandering [20]. Nevertheless, as our calculation of the drift angle does not involve any details of the core solution, we expect it to be reliable, even in the face of our inability to adequately characterize the core dynamics. In particular, our calculated drift angle is independent of both ϵ and σ. The magnitude of the drift velocity, however, is sensitive to both these parameters, and is very dependent on the details of the core. We thus expect that our theory is only qualitatively correct (see [12,13] for details) for the drift magnitude.

To confirm the above, we have performed our simulations for different values of σ. We see in Fig. 2 that the angle almost does not change with σ. This justifies that our results are generic, and essentially independent of the core, which is the aspect of the solution most sensitive to the diffusivity of the slow variable.

Obviously, the angle also depends on the amplitude of the field. Our simulations show that the dependence of the angle has some tendency to saturate as the field grows (see Fig. 3). However, above some critical value of the field the spiral does not exist.

In summary, we have calculated the velocity and the angle of the drift of the spiral resulting from the interaction with the external field. Our results are universal in the limit of small ϵ, σ and do not depend on the particular characteristics of the excitable media. Numerical simulations revealed good qualitative agreement for the angle of the drift. Experimental verification of our results is encouraged.

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